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ON SOME AFFINE ISOPERIMETRIC INEQUALITIES

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In [15] it was shown that a certain intermediary inequality can be combined with the Blaschke-Santaló inequality to obtain a general version of the affine isoperimetric inequality (of affine differential geometry) and, in turn, that the equality conditions of this intermediary inequality can be used to obtain the Blaschke-Santaló inequality if one starts with this general version of the affine isoperimetric inequality. It was shown in [16] that another intermediary inequality can be combined with the Petty projection inequality to obtain a general version of the Busemann-Petty centroid inequality and, in turn, the equality conditions of this intermediary inequality can be used to obtain the Petty projection inequality if one starts with the general version of the Busemann-Petty centroid inequality. The two situations are remarkably similar. The similarity between the Blaschke-Santaló inequality and the Petty projection inequality is striking. However, no similar analogy appears to exist between the affine isoperimetric inequality and the Busemann-Petty centroid inequality. One of the objects of this article is to show that such an analogy does exist.

The setting for this article is Euclidean n-dimensional space, $\mathbf{R}^n$ ($n \geqslant 2$). We use $\mathscr{K}^n$ to denote the space of convex bodies (compact, convex sets with nonempty interiors) in $\mathbf{R}^n$, endowed with the topology induced by the Hausdorff metric. The support function of a convex body K will be denoted by h_K; i.e.,

$$h_K(x) = \operatorname{Max}\{x \cdot y\colon\ y \in K\},$$

where $x \cdot y$ is the usual inner product of x and y in $\mathbf{R}^n$. We will usually be concerned with the restriction of h_K to the unit sphere, S^{n-1}, in $\mathbf{R}^n$. The volume of a convex body K will be denoted by $V(K)$, and for the volume of the unit ball in $\mathbf{R}^n$ we use ω_n.

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Two important points associated with a convex body K are its centroid, $\mathrm{Cen}(K)$, and its Santaló point, $\mathrm{San}(K)$. There are several equivalent definitions of the Santaló point (see [15] for a discussion). The Santaló point of K can be defined [20] as the unique point s in the interior of K such that:

$$(1) \qquad \int_{S^{n-1}} u\, h_{-s+K}^{-(n+1)}(u)\, dS(u) = 0,$$

where $dS(u)$ denotes the $(n-1)$-dimensional volume element on S^{n-1} at the point u. We recall that if K is centrally symmetric, then its Santaló point, centroid, and center of symmetry coincide.

If K is a convex body that contains the origin in its interior, then the polar body of K (with respect to the unit sphere centered at the origin) will be denoted by K^*. For an arbitrary convex body K (not necessarily containing the origin in its interior) we will use K^s to denote the polar body of K with the Santaló point of K taken as the origin; i.e.,

$$K^s = \mathrm{San}(K) + (-\mathrm{San}(K) + K)^*.$$

We shall use K^c to denote the polar body of K with the centroid of K taken as the origin; i.e.,

$$K^c = \mathrm{Cen}(K) + (-\mathrm{Cen}(K) + K)^*.$$

Rather than writing $(K^s)^c$ and $(K^c)^s$ we will simply write K^{sc} and K^{cs}. It is important (see, for example, [11], [15], [20]) that for an arbitrary convex body K one has:

$$\mathrm{Cen}(K^s) = \mathrm{San}(K) \quad \text{and} \quad \mathrm{San}(K^c) = \mathrm{Cen}(K).$$

From this observation it follows that

$$K^{sc} = K \quad \text{and} \quad K^{cs} = K.$$

A convex body A is said to have a positive continuous curvature function (see [3])

$$f_A\colon S^{n-1} \to (0, \infty),$$

provided that, for each convex body K, the mixed volume $V_1(A, K) = V(A, \cdots, A, K)$ has the integral representation

$$(2) \qquad V_1(A, K) = \frac{1}{n} \int_{S^{n-1}} f_A(u) h_K(u)\, dS(u).$$

A convex body can have at most one curvature function (see [3, p. 115]). Throughout, we use A to denote a convex body that has a positive continuous curvature function. In [22] Petty extends the classical definition of affine surface area to the class of convex bodies with positive continuous curvature function. Following Petty we define the affine surface area of A, $\Omega(A)$, by

$$\Omega(A) = \int_{S^{n-1}} f_A^{n/(n+1)}(u)\, dS(u).$$

The brightness, $\sigma_K(u)$, of a convex body K, in the direction $u \in S^{n-1}$, is the $(n-1)$-dimensional volume of the orthogonal projection of K onto the hyperplane orthogonal to u. If the convex body A has a positive continuous curvature function f_A, then it follows from the mixed volume representation of $\sigma_A(u)$ (see [3, p. 45]) that:

$$\sigma_A(u) = \frac{1}{2}\int_{S^{n-1}} |u \cdot v| f_A(v)\, dS(v). \tag{3}$$

The projection body, ΠK, of a convex body K is the convex body whose (restricted) support function is $\sigma_K(u)$; i.e.,

$$h_{\Pi K} = \sigma_K.$$

We shall write $\Pi^* K$ rather than $(\Pi K)^*$. (For a recent survey regarding projection bodies see Schneider and Weil [27].)

If K is a convex body in $\mathbf{R}^n$ and p is a point in the interior of K, then the centroid body of K, with respect to p, $\Gamma_p K$, is defined by

$$h_{\Gamma_p K}(x) = \frac{1}{V(K)}\int_{-p+K} |x \cdot y|\, dy,$$

where dy is the n-dimensional volume element at the point y. The centroid body of K with respect to the centroid of K will be denoted by ΓK; i.e., $\Gamma K = \Gamma_c K$, where $c = \mathrm{Cen}(K)$. When we refer to the centroid body of K, without specifying a particular point, we will always mean with respect to the centroid of K. If K is positioned so that its centroid is at the origin, then $h_{\Gamma K}$ can be represented (see, for example, [10, p. 250]) by:

$$h_{\Gamma K}(u) = \frac{1}{(n+1)V(K)}\int_{S^{n-1}} |u \cdot v| h_{K^*}^{-(n+1)}(v)\, dS(v). \tag{4}$$

We recall that if f is a positive continuous function on S^{n-1} such that

$$\int_{S^{n-1}} u f(u)\, dS(u) = 0,$$

then the solution of the n-dimensional Minkowski problem (see [3], [23]) guarantees the existence of a convex body (unique up to translation) whose curvature function is f. This fact, in conjunction with (1), allows us to conclude that, corresponding to an arbitrary convex body K, there is a convex body ΛK (unique up to translation) whose (positive continuous) curvature function is given by:

$$f_{\Lambda K} = \frac{V(K)}{V(K^s)} h_{-s+K}^{-(n+1)}, \tag{5}$$

where $s = \mathrm{San}(K)$. We shall refer to ΛK as the curvature image of K. (We note that our definition differs considerably from the usual definition of 'Krümmungsbild' [2].) Since ΛK is defined only up to translation, we could fix ΛK by a requirement such as $\mathrm{Cen}(\Lambda K) = 0$. We note that translating K would leave its image under Λ unchanged; i.e., $\Lambda(x + K) = \Lambda K$. If K is dilated by a factor $\lambda > 0$, then its image under Λ is also dilated by the same factor; i.e., $\Lambda(\lambda K) = \lambda \Lambda K$. For the sake of simplicity, we shall write ΛK^c rather than $\Lambda(K^c)$.

The Blaschke-Santaló inequality [1], [22], [25] (see also [24]) is:

Theorem A. *If K is a convex body in $\mathbf{R}^n$, then*

$$V(K)V(K^s) \leqslant \omega_n^2,$$

with equality if and only if K is an ellipsoid.

If we apply the Hölder integral inequality (see, for example, [10, p. 88]) to the functions $(f_A h_{-s+K})^{n/(n+1)}$ and $h_{-s+K}^{-n/(n+1)}$, where $s = \mathrm{San}(K)$ and use (2) we get:

Lemma AB. *If K and A are convex bodies in $\mathbf{R}^n$ and A has positive continuous curvature function, then*

$$\Omega(A)^{n+1} \leqslant n^{n+1} V_1^n(A, K) V(K^s),$$

with equality if and only if A and ΛK are homothetic.

Lemma AB is a (slightly reformulated) version of an inequality obtained in [15].

If Theorem A and Lemma AB are combined (see [15]), then a generalized version of the affine isoperimetric inequality (of affine differential geometry) is obtained:

Theorem B. *If K and A are convex bodies in $\mathbf{R}^n$ and A has positive continuous curvature function, then*

$$V(K)\Omega(A)^{n+1} \leqslant \omega_n^2 n^{n+1} V_1^n(A, K),$$

with equality if and only if A and K are homothetic ellipsoids.

The special case of Theorem B with $K = A$ is the extended version of the affine isoperimetric inequality [2], [25] obtained by Petty in [22].

If, in turn, we were to start with Theorem B, then by taking A to be ΛK, and observing that by Lemma AB this would mean that

$$\Omega(\Lambda K)^{n+1} = n^{n+1} V_1^n(\Lambda K, K) V(K^s),$$

we would immediately obtain Theorem A.

This relationship between Theorems A and B was shown in [15].

The Petty projection inequality [20] (see [16] for an alternate proof) is:

Theorem C. *If K is a convex body in* $\mathbf{R}^n$, *then*

$$V(K)^{n-1}V(\Pi^*K) \leqslant (\omega_n/\omega_{n-1})^n,$$

with equality if and only if K is an ellipsoid.

This inequality was used in [14] to obtain a brightness-volume inequality analogous to some width-volume inequalities of Chakerian [6], [7], [8], Chakerian and Sangwine-Yager [9] and Lutwak [13] (see also [4, pp. 170–171]). It was also used in [14] to obtain a strengthened version of the circumscribing cylinders inequality of Chakerian [6] and Knothe [12]. It was used in [16], [17] to obtain strengthened versions of the classical inequalities between the projection measures (Quermassintegrale) of convex bodies. A surprising application of the Petty projection inequality to a problem in stochastic geometry was recently found by Schneider [26].

The following (slightly reformulated) consequence of the Hölder integral inequality was obtained in [16]:

Lemma CD. *If K and* $\bar{K}$ *are convex bodies in* $\mathbf{R}^n$, *then*

$$V(\bar{K}) \leqslant ((n+1)/2)^n V_1^n(K, \Gamma\bar{K}) V(\Pi^*K),$$

with equality if and only if $\bar{K}$ *is homothetic to* Π^*K.

If Theorem C and Lemma CD are combined (see [16]) a general version of the Busemann-Petty centroid inequality is obtained:

Theorem D. *If K and* $\bar{K}$ *are convex bodies in* $\mathbf{R}^n$, *then*

$$V(K)^{n-1}V(\bar{K}) \leqslant ((n+1)\omega_n/2\omega_{n-1})^n V_1^n(K, \Gamma\bar{K}),$$

with equality if and only if K and $\bar{K}$ *are homothetic ellipsoids.*

The Busemann-Petty centroid inequality [18] is the special case $K = \Gamma\bar{K}$ of Theorem D. It was used (in a different form) by Busemann [5] to obtain his concurrent cross-section inequality. It was used by Petty [19], [20], [22] to obtain a number of important geometric inequalities.

In turn, if we were to start with Theorem D, then by taking $\bar{K}$ to be Π^*K (note that this would yield equality in the inequality of Lemma CD) in Theorem D and using Lemma CD we would obtain Theorem C. This relationship between Theorems C and D was shown in [16].

We now proceed to new material.

If we take ΠK for K in Lemma AB we obtain:

Lemma CD*. *If K and A are convex bodies in* $\mathbf{R}^n$ *and A has positive continuous curvature function, then*

$$\Omega(A)^{n+1} \leqslant n^{n+1}V_1^n(A, \Pi K)V(\Pi^*K),$$

with equality if and only if A and $\Lambda\Pi K$ *are homothetic.*

If we combine Theorem C with Lemma CD* we obtain:

Theorem D*. *If K and A are convex bodies in $\mathbf{R}^n$ and A has a positive continuous curvature function, then*

$$V(K)^{n-1}\Omega(A)^{n+1} \leqslant n(n\omega_n/\omega_{n-1})^n V_1^n(A, \Pi K),$$

with equality if and only if A and K^s are homothetic ellipsoids.

To obtain the equality conditions note that equality in the inequality of Theorem D* can occur if and only if we have equality in the inequalities of Theorem C and Lemma CD*. This is possible if and only if K is an ellipsoid and A and $\Lambda\Pi K$ are homothetic. Suppose we have equality in the inequality of Theorem D*. It is easy to verify that if K is an ellipsoid, then ΠK is an ellipsoid homothetic to K^s. The desired conclusion can now be obtained from the fact that the image under Λ of an ellipsoid is an ellipsoid homothetic to the original ellipsoid (see Lemma 2 below). The sufficiency of the equality conditions can be obtained by a similar argument.

The special case of Theorem D*, where $K = \Pi A$, is the affine projection inequality of Petty [19] (see also [22]), which states that if A is a convex body in $\mathbf{R}^n$ with positive continuous curvature function, then

$$\Omega(A)^{n+1} \leqslant n(n\omega_n/\omega_{n-1})^n V(\Pi A),$$

with equality if and only if A is an ellipsoid. That this inequality is a special case of Theorem D* follows immediately from the observation (see Lemma 6) that $V_1(A, \Pi^2 A) = V(\Pi A)$, where $\Pi^2 A = \Pi(\Pi A)$. Petty derived his affine projection inequality by using an extended version of the Busemann-Petty centroid inequality [18] which is valid for sets that are not necessarily convex. The preceding shows that the affine projection inequality of Petty can be obtained without appealing to a nonconvex version of the Busemann-Petty centroid inequality. The original motivation for much of the work presented in this article came from an attempt by the author to prove the affine projection inequality of Petty without appealing to the extended nonconvex version of the Busemann-Petty centroid inequality.

If in turn we were to start with Theorem D*, then by taking $\Lambda\Pi K$ for A in Theorem D* (which would yield equality in Lemma CD*) and using Lemma CD* we would obtain Theorem C.

Although Theorems D and D* and Lemmas CD and CD* appear to be quite different, this is not the case. If the bodies are suitably chosen, then Theorem D turns out to be a special case of Theorem D*, while Lemma CD turns out to be a special case of Lemma CD*.

From work presented in [15] and [16] it appeared likely that the analogy that exists between the Petty projection inequality and the Blaschke-Santaló inequality would also exist between the Busemann-Petty centroid inequality and the affine isoperimetric inequality. While the analogy between Theorems A and C is striking (see [14]), Theorems B and D appear to be quite different. On the other hand, the analogy between Theorems B and D* is as striking as that between Theorems A and C.

To show that Theorem D is a special case of Theorem D* and that Lemma CD is a special case of Lemma CD* we require some preliminary results.

We first observe that if A and $\bar{A}$ are convex bodies (with positive continuous curvature functions), then $f_A/f_{\bar{A}}$ is constant if and only if A and $\bar{A}$ are homothetic (see [3, p. 115]). The following lemma is easily obtained from this observation:

Lemma 1. *If K and $\bar{K}$ are convex bodies in $\mathbf{R}^n$, then ΛK is homothetic to $\Lambda \bar{K}$ if and only if K is homothetic to $\bar{K}$.*

It is easy to verify that the curvature function of an ellipsoid E with center e is a constant multiple of $h_{-e+E}^{-(n+1)}$. The following lemma is a direct consequence of this fact and the observation above:

Lemma 2. *If ΛK is an ellipsoid, then K is an ellipsoid homothetic to ΛK; conversely, if K is an ellipsoid, then ΛK is an ellipsoid homothetic to K.*

Another result we require is:

Lemma 3. *If K is a convex body in $\mathbf{R}^n$, then*

$$V_1(\Lambda K, K) = V(K).$$

Proof. From the translation invariance of mixed volumes it follows that $V_1(\Lambda K, K) = V_1(\Lambda K, -s + K)$, where $s = \operatorname{San}(K)$. From (2) we have

$$V_1(\Lambda K, K) = \frac{1}{n}\int_{S^{n-1}} f_{\Lambda K}(u) h_{-s+K}(u)\, dS(u).$$

Hence, it follows from (5) that

$$V(K^s)V(K)^{-1}V_1(\Lambda K, K) = \frac{1}{n}\int_{S^{n-1}} h_{-s+K}^{-n}(u)\, dS(u).$$

The desired result is, now, obtained by observing that the integral on the right is just $V(K^s)$.

We observe that from Lemma 3 it follows that if ΛK is homothetic to K, then ΛK must, in fact, be a translate of K.

As an aside we note that, by using the Minkowski inequalities [3, p. 91] in conjunction with Lemma 3, one can conclude that if K is a convex body in $\mathbf{R}^n$, then:

$$V(\Lambda K) \leqslant V(K),$$

with equality if and only if ΛK is a translation of K.

For the affine surface area of ΛK one has:

Lemma 4. *If K is a convex body in $\mathbf{R}^n$, then*

$$\Omega(\Lambda K)^{n+1} = n^{n+1} V(K)^n V(K^s).$$

The proof involves a routine computation using the definition of affine surface area, the polar coordinate formula for the volume of K^s and (5). (Lemma 4 can also be obtained from the equality conditions of Lemma AB.)

Results almost identical to those of Lemmas 3 and 4 can be found in, for example, [1], [21], [22], [25].

As an aside, we observe that, by using the affine isoperimetric inequality in conjunction with Lemmas 2 and 4, we can conclude that if K is a convex body in $\mathbf{R}^n$, then

$$V(K)^n V(K^s) \leqslant \omega_n^2 V(\Lambda K)^{n-1},$$

with equality if and only if K is an ellipsoid. If we were to combine this with the fact (noted earlier) that $V(\Lambda K) \leqslant V(K)$, the result would be the Blaschke-Santaló inequality.

A useful factorization of the centroid operator is given in:

Lemma 5. *If K is a convex body in $\mathbf{R}^n$, then*

$$\Gamma K = \mu \Pi \Lambda K^c,$$

where $\mu = 2/(n+1)V(K^c)$.

Proof. There is no loss of generality in assuming that K is positioned so that its centroid is at the origin. From (4) we have

$$h_{\Gamma K}(u) = \frac{1}{(n+1)V(K)} \int_{S^{n-1}} |u \cdot v| h_K^{-(n+1)}(v)\, dS(v).$$

If we use (5), the fact that $\operatorname{San}(K^c) = \operatorname{Cen}(K) = 0$, and that $V(K^{cs}) = V(K)$, we have

$$h_{\Gamma K}(u) = \frac{1}{(n+1)V(K^c)} \int_{S^{n-1}} |u \cdot v| f_{\Lambda K^c}(v)\, dS(v).$$

From the definition of projection body and (3) we can now obtain the desired result:

$$h_{\Gamma K}(u) = \frac{2}{(n+1)V(K^c)} h_{\Pi \Lambda K^c}(u).$$

The conclusion of Lemma 5 is very similar to that of Theorem (3.11) in [21].

Lemma 6. *If K and $\bar{K}$ are convex bodies in $\mathbf{R}^n$, then*

$$V_1(K, \Pi \bar{K}) = V_1(\bar{K}, \Pi K).$$

Proof. We first assume that K and $\bar{K}$ have positive continuous curvature functions. From (2) we have

$$V_1(K, \Pi\bar{K}) = \frac{1}{n}\int_{S^{n-1}} f_K(u)\sigma_{\bar{K}}(u)\,dS(u).$$

From (3) we obtain

$$V_1(K, \Pi\bar{K}) = \frac{1}{2n}\int_{S^{n-1}} f_K(u)\int_{S^{n-1}} |u\cdot v| f_{\bar{K}}(v)\,dS(v)\,dS(u).$$

If we change the order of integration and use (3) we get

$$V_1(K, \Pi\bar{K}) = V_1(\bar{K}, \Pi K).$$

When we combine this result with a standard approximation argument (that makes use of the continuity of the mixed volumes and the fact that the set of convex bodies with positive continuous curvature functions is dense in $\mathscr{K}^n$) we obtain the desired result for arbitrary convex bodies K and $\bar{K}$ in $\mathbf{R}^n$.

If we combine the results of Lemmas 5 and 6 we get:

Lemma 7. *If K and $\bar{K}$ are convex bodies in* $\mathbf{R}^n$, *then*

$$V_1(\Lambda K^c, \Pi\bar{K}) = ((n+1)/2)V(K^c)V_1(\bar{K}, \Gamma K).$$

We are now in a position to show that Theorem D is a special case of Theorem D*.

Suppose that K and $\bar{K}$ are arbitrary convex bodies in $\mathbf{R}^n$. If we take A to be $\Lambda\bar{K}^c$ in Theorem D* we have:

$$V(K)^{n-1}\Omega(\Lambda\bar{K}^c)^{n+1} \leqslant n(n\omega_n/\omega_{n-1})^n V_1^n(\Lambda\bar{K}^c, \Pi K). \tag{6}$$

with equality if and only if $\Lambda\bar{K}^c$ and K^s are homothetic ellipsoids. From Lemma 4, and the fact that $\bar{K}^{cs} = \bar{K}$, we have

$$\Omega(\Lambda\bar{K}^c)^{n+1} = n^{n+1}V(\bar{K}^c)^n V(\bar{K}).$$

Hence, by using Lemma 7, we can rewrite (6) as

$$V(K)^{n-1}V(\bar{K}) \leqslant ((n+1)\omega_n/2\omega_{n-1})^n V_1^n(K, \Gamma\bar{K}),$$

which is the inequality of Theorem D. To obtain the equality conditions note that from Lemma 2 it follows that $\Lambda\bar{K}^c$ and K^s are homothetic ellipsoids if and only if $\bar{K}^c$ and K^s are homothetic ellipsoids, or, equivalently, if and only if $\bar{K}$ and K are homothetic ellipsoids. This is precisely the condition for equality in Theorem D.

To see that Lemma CD is a special case of Lemma CD*, suppose that K and $\bar{K}$ are arbitrary convex bodies in $\mathbf{R}^n$. If we take A to be $\Lambda\bar{K}^c$ in Lemma CD* we have:

$$\Omega(\Lambda\bar{K}^c)^{n+1} \leqslant n^{n+1}V_1^n(\Lambda\bar{K}^c, \Pi K)V(\Pi^* K), \tag{7}$$

with equality if and only if $\Lambda\bar{K}^c$ and $\Lambda\Pi K$ are homothetic. As above, we can rewrite (7) as

$$V(\bar{K}) \leqslant ((n+1)/2)^n V_1^n(K, \Gamma\bar{K}) V(\Pi^* K),$$

which is the inequality of Lemma CD. To obtain the equality conditions note that, from Lemma 1, it follows that $\Lambda\bar{K}^c$ and $\Lambda\Pi K$ are homothetic if and only if $\bar{K}^c$ and ΠK are homothetic, or, equivalently, if and only if $\bar{K}^{cs} = \bar{K}$ and $\Pi^* K$ (since ΠK is centrally symmetric) are homothetic. This is the condition for equality in Lemma CD.

The methods used to show that Theorem D is a special case of Theorem D* can also be used to derive other new inequalities:

Theorem E. *If K and $\bar{K}$ are convex bodies in $\mathbf{R}^n$, then*

$$V(\Pi\bar{K}) V(K) \leqslant ((n+1)/2)^n \omega_n^2 V_1^n(\bar{K}, \Gamma K).$$

with equality if and only if K and $\Pi^*\bar{K}$ are homothetic ellipsoids.

Proof. If we take $\Pi\bar{K}$ for K and ΛK^c for A in Theorem B, and use Lemmas 4 and 7 (as above), we obtain:

$$V(\Pi\bar{K}) V(K) \leqslant ((n+1)/2)^n \omega_n^2 V_1^n(\bar{K}, \Gamma K),$$

with equality if and only if ΛK^c and $\Pi\bar{K}$ are homothetic ellipsoids. However, ΛK^c and $\Pi\bar{K}$ are homothetic ellipsoids if and only if (by Lemma 2) K^c and $\Pi\bar{K}$ are homothetic ellipsoids, or, equivalently, if and only if K and $\Pi^*\bar{K}$ are homothetic ellipsoids.

If we take $\Pi\bar{K}$ for K in Theorem E we obtain:

Corollary E1. *If K is a convex body in $\mathbf{R}^n$, then*

$$V(\Pi K)^2 \leqslant ((n+1)/2)^n \omega_n^2 V_1^n(K, \Gamma\Pi K).$$

with equality if and only if K is a body of constant brightness.

To obtain the conditions for equality note that from Theorem E it follows that equality in the inequality of Corollary E1 is possible if and only if ΠK and $\Pi^* K$ are homothetic ellipsoids. But the polar body of an ellipsoid (with respect to its center) is homothetic to the original ellipsoid if and only if the original ellipsoid is a ball. Hence, equality implies that ΠK is a ball or, equivalently, that K is a body of constant brightness.

If we take ΓK for $\bar{K}$ in Theorem E, we obtain:

Corollary E2. *If K is a convex body in $\mathbf{R}^n$, then*

$$V(\Pi\Gamma K) V(K) \leqslant ((n+1)/2)^n \omega_n^2 V(\Gamma K)^n,$$

with equality if and only if K is an ellipsoid.

Throughout we have chosen to deal with the centroid body of K with respect to the centroid of K rather than with respect to some arbitrary point in the interior of K. Although this appears to be a less general treatment, this is not the case. Theorem D as stated earlier is not the actual version given in [16]; the actual version is:

Theorem D$^+$. *If K and $\bar{K}$ are convex bodies in $\mathbf{R}^n$, and p is a point in the interior of $\bar{K}$, then*

$$V(K)^{n-1}V(\bar{K}) \leqslant ((n+1)\omega_n/2\omega_{n-1})^n V_1^n(K, \Gamma_p\bar{K}),$$

with equality if and only if K and $\bar{K}$ are homothetic ellipsoids and $\bar{K}$ has center p.

The fact is that once Theorem D is established, then Theorem D$^+$ can be obtained as a simple consequence. In fact, Theorem D$^+$ can be obtained from the following much weaker version (of Theorem D):

Theorem D$^-$. *If K is a centrally symmetric convex body in $\mathbf{R}^n$, then*

$$V(K) \leqslant ((n+1)\omega_n/2\omega_{n-1})^n V(\Gamma K).$$

with equality if and only if K is an ellipsoid.

To see that Theorem D$^+$ can be obtained from Theorem D$^-$, suppose K is an arbitrary convex body in $\mathbf{R}^n$. We first observe that from the equality conditions of Lemma CD (or by direct computation) one has the identity:

$$V_1(K, \Gamma\Pi^*K) = 2/(n+1).$$

If we combine this with the Minkowski inequalities [3, p. 91] we obtain:

$$V(K)^{n-1}V(\Gamma\Pi^*K) \leqslant (2/(n+1))^n, \tag{8}$$

with equality if and only if K and $\Gamma\Pi^*K$ are homothetic. Since Π^*K is centrally symmetric, it follows from Theorem D$^-$ that:

$$V(K)^{n-1}V(\Pi^*K) \leqslant (\omega_n/\omega_{n-1})^n, \tag{9}$$

with equality if and only if K is an ellipsoid. The necessity of the equality conditions in (9) follows from the fact that equality in (9) would imply (from the equality conditions in (8)) that K and $\Gamma\Pi^*K$ are homothetic, and (from the equality conditions of Theorem D$^-$) that Π^*K is in fact an ellipsoid. Hence, from the fact that K and $\Gamma\Pi^*K$ are homothetic (and Π^*K is an ellipsoid), it follows that K must be an ellipsoid.

Thus, from Theorem D$^-$, we can obtain the Petty projection inequality (Theorem C). We now show that Theorem D$^+$ follows from the Petty projection inequality.

Lemma CD as previously stated is a reformulated version of Corollary (5.6) of [16]; the version given in [16] is equivalent to:

Lemma CD$^+$. *If K and $\bar{K}$ are convex bodies in $\mathbf{R}^n$, and p is a point in the interior of $\bar{K}$, then*

$$V(\bar{K}) \leqslant ((n+1)/2)^n V_1^n(K, \Gamma_p\bar{K}) V(\Pi^*K),$$

with equality if and only if $\bar{K}$ is homothetic to Π^*K and has center p.

We note that Lemma CD^+ is a simple consequence of the Hölder integral inequality [10, p. 88] (for details see [16]).

If we combine Lemma CD^+ with the Petty projection inequality we can conclude that if K and $\bar{K}$ are convex bodies in $\mathbf{R}^n$ and p is a point in the interior of $\bar{K}$, then:

$$V(K)^{n-1}V(\bar{K}) \leqslant ((n+1)\omega_n/2\omega_{n-1})^n V_1^n(K, \Gamma_p\bar{K}), \tag{10}$$

with equality if and only if K and $\bar{K}$ are homothetic ellipsoids and $\bar{K}$ has center p. To obtain the necessity of the equality conditions note that equality in (10) would imply that K is an ellipsoid (from the equality conditions in the Petty projection inequality) and that $\bar{K}$ is homothetic to Π^*K and has center p (from the equality conditions in Lemma CD^+). But if K is an ellipsoid, then Π^*K is an ellipsoid homothetic to K (see, for example, [16, p. 102]). A similar argument can be used to prove the sufficiency of the equality conditions. We have thus obtained Theorem D^+ from the Petty projection inequality.

The preceding also shows that once one obtains the Busemann-Petty centroid inequality for the class of centrally symmetric bodies (or even just for polars of projection bodies), then the Busemann-Petty centroid inequality (valid for all convex bodies) follows easily.

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